

The Coulomb Branch Formula for Quiver Moduli Spaces

Jan Manschot¹, Boris Pioline^{2,3,4}, Ashoke Sen⁵

¹ *Institut Camille Jordan, Université Claude Bernard Lyon 1,*

43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France

² *CERN PH-TH, Case C01600, CERN, CH-1211 Geneva 23, Switzerland*

³ *Sorbonne Universités, UPMC Univ. Paris 06, UMR 7589, LPTHE, F-75005, Paris, France*

⁴ *CNRS, UMR 7589, LPTHE, F-75005, Paris, France*

⁵ *Harish-Chandra Research Institute, Chhatnag Road, Jhusi, Allahabad 211019, India*

email: jan.manschot@univ-lyon1.fr, boris.pioline@cern.ch, sen@hri.res.in

Abstract

In recent series of works, by translating properties of multi-centered supersymmetric black holes into the language of quiver representations, we proposed a formula that expresses the Hodge numbers of the moduli space of semi-stable representations of quivers with generic superpotential in terms of a set of invariants associated to ‘single-centered’ or ‘pure-Higgs’ states. The distinguishing feature of these invariants is that they are independent of the choice of stability condition. Furthermore they are uniquely determined by the χ_g -genus of the moduli space. Here, we provide a self-contained summary of the Coulomb branch formula, spelling out mathematical details but leaving out proofs and physical motivations.

Contents

1	Introduction	2
2	Statement of the Coulomb branch formula	5
2.1	The main formula	6
2.2	The Coulomb index g_C	6
2.2.1	First definition	7
2.2.2	Second definition	8
2.3	Definition of the functions H	11
2.4	Constraints on $\Omega_S(\alpha; t)$'s	12
2.5	Deformations of the input parameters	13
2.6	Mathematica code	15
3	Aspects of the Coulomb branch formula	16
3.1	Quivers without oriented loops and Reineke's formula	16
3.2	Wall-crossing	17
3.3	Generalized quiver mutations	18
3.4	Other tests	20
3.5	Variants of the conjecture	21
3.6	Application to framed BPS indices	22

1 Introduction

A quiver is a collection of nodes $i = 1 \dots K$ connected by arrows $i \longrightarrow j$, and decorated by positive integers N_i and real parameters ζ_i . In the physics literature, quivers describe the matter content of supersymmetric gauge theories [1, 2]. The nodes label $U(N_i)$ gauge groups, the arrows denote bifundamental matter, while the ζ_i 's are known as Fayet-Iliopoulos (FI) parameters. In the mathematics literature, quivers are fundamental objects in representation theory (see e.g. [3, 4] for entry points). The collection of integers (N_1, \dots, N_K) is called the dimension vector and the parameters $(\zeta_1, \dots, \zeta_K)$ define a choice of stability condition [5].

We restrict to 2-acyclic quivers, i.e. such that there are no arrows running in both directions between any two nodes – in a generic situation such arrows can be removed in pairs without affecting the relevant results. Let γ_{ij} be the number of arrows from the i -th node to the j -th node: γ_{ij} is a positive integer if the arrows are directed from the i -th to the j -th node, and

negative in the opposite situation. There is no loss of generality in assuming that the FI parameters associated to any quiver satisfy

$$\sum_{i=1}^K N_i \zeta_i = 0. \quad (1.1)$$

It is convenient to introduce an abstract K -dimensional vector space spanned by basis elements $\gamma_1, \dots, \gamma_K$, and associate a vector $\gamma = \sum_i N_i \gamma_i$ to a quiver with dimension vector $N \equiv (N_1, \dots, N_K)$. Since N_i 's are integers, the vector γ belongs to a K -dimensional lattice Γ spanned by basis vectors γ_i . We also denote by Γ^+ the cone of lattice vectors of the form $\sum_i n_i \gamma_i$ with $n_i \geq 0$; all physical quivers are described by some vector $\gamma \in \Gamma^+$. Finally we introduce a vector space \mathcal{V} consisting of elements of the form $\sum_i u_i \gamma_i$ with $u_i \in \mathbb{R}$, and a bilinear symplectic product (the Dirac-Schwinger-Zwanziger, or DSZ product) on \mathcal{V} via

$$\left\langle \sum_i u_i \gamma_i, \sum_j v_j \gamma_j \right\rangle = \sum_{i,j} \gamma_{ij} u_i v_j. \quad (1.2)$$

It will also be convenient to introduce the inner product between the FI parameters $\zeta = (\zeta_1, \dots, \zeta_K)$ and a vector $u = \sum_i u_i \gamma_i \in \mathcal{V}$:

$$\zeta \cdot u = \sum_{i=1}^K \zeta_i u_i, \quad (1.3)$$

so that eq.(1.1) can be expressed as $\zeta \cdot \gamma = 0$. In the following we shall denote by $\mathcal{Q}(\gamma; \zeta)$ a quiver with dimension vector N and FI parameters ζ satisfying (1.1).

To each quiver $\mathcal{Q}(\gamma; \zeta)$, one associates a quiver moduli space $\mathcal{M}(\gamma; \zeta)$ defined as follows. One introduces complex variables $\phi_{\ell k, \alpha, ss'}$ for every pair of nodes ℓ, k for which $\gamma_{\ell k} > 0$. Here α runs over $\gamma_{\ell k}$ values, s is an index labelling the fundamental representation of $U(N_\ell)$ and s' is an index representing the anti-fundamental representation of $U(N_k)$. The quiver moduli space $\mathcal{M}(\gamma; \zeta)$ is defined to be the space of solutions to the D-term and F-term constraints,

$$\begin{aligned} \sum_{\substack{k, s, t, s', \alpha \\ \gamma_{\ell k} > 0}} \phi_{\ell k, \alpha, ss'}^* T_{st}^{(\ell)a} \phi_{\ell k, \alpha, ts'} - \sum_{\substack{k, s, t, s', \alpha \\ \gamma_{k\ell} > 0}} \phi_{k\ell, \alpha, s't}^* T_{st}^{(k)a*} \phi_{k\ell, \alpha, st} &= \zeta_\ell \text{Tr}(T^{(\ell)a}) \quad \forall \ell, a, \\ \frac{\partial W}{\partial \phi_{\ell k, \alpha, ss'}} &= 0, \quad \forall \ell, k, \alpha, s, s', \end{aligned} \quad (1.4)$$

modded out by the natural action of the gauge group $\prod_\ell U(N_\ell)$. Here $T^{(\ell)a}$'s are the generators of the $U(N_\ell)$ gauge group, and W is a generic gauge invariant superpotential holomorphic in the

variables $\phi_{\ell k, \alpha, ss'}$. Equivalently, one may define $\mathcal{M}(\gamma; \zeta)$ as the quotient of a certain open subset of solutions of the F-term constraints (the second line in (1.4)) by the action of the complexified gauge group $\prod_{\ell} GL(N_{\ell}, \mathbb{C})$. The open subset is the space of quiver representations which are semi-stable with respect to ζ [5]. This induces a natural complex structure on $\mathcal{M}(\gamma; \zeta)$, and shows that $\mathcal{M}(\gamma; \zeta)$ is a quasi-projective algebraic variety. For generic superpotential (i.e. such that W should contain at least a generic linear combination of all the generators of the ring of gauge invariant polynomials of the variables $\phi_{\ell k, \alpha, ts}$) and primitive charge vector γ (i.e. $\gcd(\{N_i\}) = 1$), $\mathcal{M}(\gamma; \zeta)$ is believed to be compact, projective and smooth.

Let $h^{p,q}(\mathcal{M})$ denote the Hodge numbers of the quiver moduli space $\mathcal{M}(\gamma; \zeta)$. We define the Dolbeault polynomial of $\mathcal{M}(\gamma; \zeta)$ to be

$$Q(\gamma; \zeta; y, t) \equiv \sum_{p,q} h^{p,q}(\mathcal{M}) (-y)^{p+q-d} t^{p-q}, \quad (1.5)$$

where y and t are complex variables. Thus $Q(\gamma; \zeta; y, t)$ contains complete information about the Hodge numbers of $\mathcal{M}(\gamma; \zeta)$. The specification of $Q(\gamma; \zeta; y, t)$ at $y = 1/t = v^{1/2}$ is proportional to the Hirzebruch polynomial (or χ_y -genus),

$$Q(\gamma; \zeta; y = v^{1/2}; t = v^{-1/2}) \equiv \sum_{p,q} h^{p,q}(\mathcal{M}) (-1)^{p+q-d} v^{q-d/2} = (-1)^d v^{-d/2} \chi(\mathcal{M}, v) \quad (1.6)$$

where $\chi(\mathcal{M}, v) \equiv \sum_{p,q} h^{p,q}(\mathcal{M}) (-1)^{p+q} v^q$. The Hirzebruch polynomial can in principle be computed using the Hirzebruch-Riemann-Roch theorem (see *e.g.* [6]).

In order to deal with cases where γ is not primitive, it is useful to define

$$\bar{Q}(\gamma; \zeta; y, t) = \sum_{m|\gamma} \frac{1}{m} \frac{y - y^{-1}}{y^m - y^{-m}} Q(\gamma/m; \zeta; y^m, t^m), \quad (1.7)$$

where $m|\gamma$ means that m is a common divisor of (N_1, \dots, N_K) if $\gamma = \sum_{\ell} N_{\ell} \gamma_{\ell}$. $\bar{Q}(\gamma; \zeta; y, t)$ is a polynomial in t with coefficients which are rational in y . The Dolbeault polynomial $Q(\gamma; \zeta; y, t)$ can be recovered from $\bar{Q}(\gamma; \zeta; y, t)$ by the Möbius inversion formula,

$$Q(\gamma; \zeta; y, t) = \sum_{m|\gamma} \frac{\mu(m)}{m} \frac{y - y^{-1}}{y^m - y^{-m}} \bar{Q}(\gamma/m; \zeta; y^m, t^m), \quad (1.8)$$

where $\mu(m)$ is the Möbius function (i.e. 1 if m is a product of an even number of distinct primes, -1 if m is a product of an odd number of distinct primes, or 0 otherwise). For primitive γ , Q and \bar{Q} coincide. If γ is not primitive, the moduli space $\mathcal{M}(\gamma; \zeta)$ is not expected to be compact and the algebraic-geometric definition of either Q or \bar{Q} is more involved [7, 8]. The

coefficients of Q are expected to give the intersection cohomology of the compactified moduli space. Moreover, \bar{Q} is a particularly useful construct as it appears in the wall-crossing formula (3.4) even for quivers with primitive total charge.

The Coulomb branch formula proposed in [9–11] is a conjectural formula for $\bar{Q}(\gamma; \zeta; y, t)$ for an arbitrary quiver, under the assumption that the superpotential W is generic. This formula was motivated from the study of multi-centered black hole quantum mechanics in $\mathcal{N} = 2$ supergravity. Our goal in this note is to explain this formula in mathematical detail, leaving out physical motivations. For the physical reasoning that led to the formula, and for the proof of some of its properties, the reader is referred to the original papers. More pedagogical reviews can be found in [12, 13].

Finally a word about notation: throughout our discussion we shall reserve the symbols $\gamma_1, \dots, \gamma_K$ for the basis vectors of Γ^+ as introduced above. On the other hand the symbols γ , α_i , β_i etc. will be used to label more general elements of Γ^+ and occasionally also elements of the vector space \mathcal{V} which do not necessarily lie in Γ^+ .

2 Statement of the Coulomb branch formula

The Coulomb branch formula expresses $Q(\gamma; \zeta; y, t)$ in terms of a set of ‘single-centered’ or ‘pure-Higgs’ indices $\Omega_S(\alpha; t)$. Since this formula has not been proven in full generality, we shall denote it by $Q_C(\gamma; \zeta; y, t)$ with the understanding that Q_C is conjecturally equal to Q .¹ The formula will only be valid away from the walls of marginal stability, i.e. for choices of FI parameters ζ for which

$$\sum_{i=1}^K n_i \zeta_i \neq 0 \quad \text{for} \quad 0 \leq n_i \leq N_i \quad (2.1)$$

except when $n_i = q N_i \forall i$ for some rational number q . On a wall of marginal stability the moduli space \mathcal{M} and hence the associated Dolbeault polynomial are ill-defined.

¹The functions Q_C and g_C were denoted by Q_{Coulomb} and g_{Coulomb} in [11].

2.1 The main formula

The proposed formula for $\bar{Q}_C(\gamma; \zeta; y, t)$ takes the form

$$\bar{Q}_C(\gamma; \zeta; y, t) = \sum_{n \geq 1} \sum_{\substack{\{\alpha_i \in \Gamma^+\} \\ \sum_{i=1}^n \alpha_i = \gamma}} \frac{1}{n!} g_C(\{\alpha_1, \dots, \alpha_n\}, \{c_1, \dots, c_n\}; y) \prod_{i=1}^n \left\{ \sum_{\substack{m_i \in \mathbb{Z} \\ m_i | \alpha_i}} \frac{1}{m_i} \frac{y - y^{-1}}{y^{m_i} - y^{-m_i}} \Omega_{\text{tot}}(\alpha_i / m_i; y^{m_i}, t^{m_i}) \right\}, \quad (2.2)$$

where the sums over n and $\{\alpha_1, \dots, \alpha_n\}$ label all possible ways of expressing γ as *ordered* sums of elements α_i of Γ^+ . The functions $g_C(\{\alpha_1, \dots, \alpha_n\}; \{c_1, \dots, c_n\}; y)$, known as Coulomb indices, will be given explicitly in §2.2.² The c_i 's in the argument of g_C are given in terms of the FI parameters by

$$c_i = \zeta \cdot \alpha_i. \quad (2.3)$$

The condition (1.1) together with $\sum_i \alpha_i = \gamma$ guarantees that $\sum_i c_i = 0$. Finally the functions $\Omega_{\text{tot}}(\alpha; y, t)$ are given by

$$\Omega_{\text{tot}}(\alpha; y, t) = \Omega_S(\alpha; t) + \sum_{\substack{\{\beta_i \in \Gamma^+\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \sum_i m_i \beta_i = \alpha}} H(\{\beta_i\}; \{m_i\}; y) \prod_i \Omega_S(\beta_i; t^{m_i}). \quad (2.4)$$

The sum over $\{\beta_i\}$ and $\{m_i\}$ run over *unordered* sets satisfying the constraint $\sum_i m_i \beta_i = \alpha$. $H(\{\beta_i\}; \{m_i\}; y)$ are functions which can be determined from the functions g_C following an algorithm to be described in §2.3. The functions $\Omega_S(\alpha; t)$, called single centered indices, are unknown functions of t but are independent of y and ζ . They are required to satisfy $\Omega_S(\alpha; t) = 0$ for $\alpha \notin \Gamma^+$. They also have no explicit dependence on the vector γ labelling the original quiver; so for different γ 's we use the same set of $\Omega_S(\alpha; t)$'s. Some further constraints on $\Omega_S(\alpha; t)$, as well as their determination from the Hirzebruch polynomial, will be discussed in §2.4.

2.2 The Coulomb index g_C

Our goal in this subsection is to give an expression for $g_C(\{\alpha_1, \dots, \alpha_n\}, \{c_1, \dots, c_n\}; y)$. First we define

$$\alpha_{ij} = \langle \alpha_i, \alpha_j \rangle. \quad (2.5)$$

²Since this function is symmetric in the arguments $\{\alpha_1, \dots, \alpha_n\}$, we can replace the sum over the ordered set $\{\alpha_1, \dots, \alpha_n\}$ by the sum over an unordered set, and replace the $n!$ in the denominator by a symmetry factor $|\text{Aut}(\{\alpha_1, \dots, \alpha_n\})|$ given by $\prod_k s_k!$ if among the set $\{\alpha_i\}$ there are s_1 identical vectors $\tilde{\alpha}_1$, s_2 identical vectors $\tilde{\alpha}_2$ etc.

There are two equivalent definitions of g_C . The first one is simple to state but involves testing the existence of solutions of some non-linear algebraic equations. The second one is an iterative procedure that is more complicated to state but does not require solving any equations, and hence is a faster procedure. In both definitions we have, for a single charge vector,

$$g_C(\{\alpha_1\}; \{c_1 = 0\}; y) = 1 \quad \forall \alpha_1. \quad (2.6)$$

2.2.1 First definition

$$g_C(\{\alpha_1, \dots, \alpha_n\}; \{c_1, \dots, c_n\}; y) = (-1)^{n-1+\sum_{i<j} \alpha_{ij}} (y - y^{-1})^{-n+1} \sum_{\text{solutions } P} s(P) y^{\sum_{i<j} \alpha_{ij} \text{sign}(z_j - z_i)}, \quad (2.7)$$

where the sum runs over solutions to the equations

$$\sum_{\substack{j=1 \\ j \neq i}}^n \frac{\alpha_{ij}}{|z_i - z_j|} = c_i, \quad \text{for } 1 \leq i \leq n-1, \quad z_1 = 0. \quad (2.8)$$

$s(P)$ in (2.7) is given by the sign of the Hessian $\det(\partial^2 V / \partial z_i \partial z_j)$ of the function

$$V(\{z_i\}) = - \sum_{\substack{i,j=1 \\ i < j}}^n \alpha_{ij} \text{sign}(z_j - z_i) \log |z_i - z_j| - \sum_{i=1}^n c_i z_i, \quad (2.9)$$

whose critical points reproduce the conditions (2.8). This prescription follows from computing the index of the Dirac operator on the moduli space of multi-centered black holes using the Atiyah-Bott-Lefschetz formula [9, 14].

Eq.(2.7) gives an unambiguous definition of $g_C(\{\alpha_1, \dots, \alpha_n\}; \{c_1, \dots, c_n\}; y)$ when each solution to (2.8) has all the centers distinct. However the enumeration of distinct solutions becomes ambiguous when two or more centers coincide.³ The remedy found in [11] is to work with a deformed set of parameters. The rules for generating these deformations will be reviewed in §2.5.

The algorithm described above requires us to find all solutions to (2.8) and hence in general can only be carried out numerically. (Note however that the final expression (2.7) is insensitive to the details of the solution and depends only on the relative ordering of the z_i 's in a given solution). There is however an equivalent definition of $g_C(\{\alpha_1, \dots, \alpha_n\}; \{c_1, \dots, c_n\}; y)$, derived from the above definition, in which each step can be carried out analytically. We shall now give this alternative definition, but readers interested in skipping the details can jump to §2.3 for the definitions of the functions H appearing in (2.4).

³Since $1/|z_i - z_j|$ diverges as $z_i \rightarrow z_j$, to examine the existence of solutions with coincident centers we need to take $|z_i - z_j| \sim \epsilon$ for a subset of the centers and then examine if it is possible to satisfy (2.8) in the $\epsilon \rightarrow 0$ limit for suitable choice of ratios of these distances. These were called collinear scaling solutions in [11].

2.2.2 Second definition

The alternate definition of g_C takes the form

$$\begin{aligned} & g_C(\{\alpha_1, \dots, \alpha_n\}; \{c_1, \dots, c_n\}; y) \\ = & (-1)^{n-1+\sum_{i<j} \alpha_{ij}} (y - y^{-1})^{-n+1} \sum_{\sigma} F(\{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}\}; \{c_{\sigma(1)}, \dots, c_{\sigma(n)}\}) y^{\sum_{i<j} \alpha_{\sigma(i)\sigma(j)}}, \end{aligned} \quad (2.10)$$

where the sum runs over all permutations σ of $\{1, 2, \dots, n\}$. The function F is determined recursively by the equations

$$F(\{\tilde{\alpha}_1\}; \{\tilde{c}_1 = 0\}) = 1, \quad (2.11)$$

$$\begin{aligned} & F(\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_n\}; \{\tilde{c}_1, \dots, \tilde{c}_n\}) \\ = & \Theta(-\tilde{\alpha}_{n-1,n} \tilde{c}_n) (-1)^{\Theta(-\tilde{\alpha}_{n-1,n})} F(\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_{n-1}\}; \{\tilde{c}_1, \dots, \tilde{c}_{n-2}, \tilde{c}_{n-1} + \tilde{c}_n\}) \\ & + \sum_{k=0}^{n-3} F(\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_k, \tilde{\alpha}_{k+1} + \dots + \tilde{\alpha}_{n-1} + \lambda_k \tilde{\alpha}_n\}; \{\tilde{c}_1, \dots, \tilde{c}_k, \tilde{c}_{k+1} + \dots + \tilde{c}_n\}) \\ & \times G(\tilde{\alpha}_{k+1}, \dots, \tilde{\alpha}_{n-1}, \lambda_k \tilde{\alpha}_n) \Theta \left(- \sum_{\substack{i,j \\ k+1 \leq i < j \leq n-1}} \tilde{\alpha}_{ij} \sum_{\substack{i,j \\ k+1 \leq i < j \leq n}} \tilde{\alpha}_{ij} \right) \text{sign} \left(\sum_{i=k+1}^{n-1} \tilde{\alpha}_{kn} \right), \end{aligned} \quad (2.12)$$

where $\tilde{\alpha}_i$ are arbitrary elements of \mathcal{V} , and

$$\lambda_k = - \sum_{\substack{i,j \\ k+1 \leq i < j \leq n-1}} \tilde{\alpha}_{ij} / \sum_{i=k+1}^{n-1} \tilde{\alpha}_{in}. \quad (2.13)$$

$\Theta(x)$ is the step function

$$\Theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}, \quad (2.14)$$

and the functions G are determined from another set of recursion relations to be given below. Once the functions G are known, (2.12) can be used to construct F recursively starting with the initial value in (2.11).

Physically $F(\{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}\}; \{c_{\sigma(1)}, \dots, c_{\sigma(n)}\})$ represents the sum of the signs $s(P)$ introduced in §2.2.1 for all solutions P in which the z_i 's are ordered as $z_{\sigma(1)} < z_{\sigma(2)} < \dots < z_{\sigma(n)}$. $G(\{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}\})$ has a very similar interpretation except that in obtaining solutions to (2.8) we set all the c_i 's to zero. A useful property of G that follows from this is that $G(\hat{\alpha}_1, \dots, \hat{\alpha}_n)$ for

any $\{\hat{\alpha}_i\}$ vanishes if the set $\{1, \dots, n\}$ can be divided into two sets A and B such that $\hat{\alpha}_{ij} > 0$ for all $i \in A$ and $j \in B$ [15].

Note that the step and sign functions appearing on the right hand side of (2.12) and similar factors appearing in the recursion relations for G to be described below, will be ambiguous if their argument vanishes. The remedy given in [11] is again to work with a deformed set of parameters. This will be reviewed in §2.5.

We now turn to the definition of the function G . First of all $G(\hat{\alpha}_1, \dots, \hat{\alpha}_n)$ is non-zero only if $\sum_{i < j} \hat{\alpha}_{ij} = 0$. Moreover G vanishes if $n < 3$. For $n = 3$,

$$G(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) = \Theta(\hat{\alpha}_{12}\hat{\alpha}_{23}) (-1)^{\Theta(\hat{\alpha}_{23})+1}, \quad \hat{\alpha}_i \in \mathcal{V}. \quad (2.15)$$

Another important property of G is that it depends on its arguments $\hat{\alpha}_i$ only via the DSZ products $\hat{\alpha}_{ij} = \langle \hat{\alpha}_i, \hat{\alpha}_j \rangle$. Hence in the recursion relations below, it is sufficient to specify at every stage the $\hat{\alpha}_{ij}$'s appearing in the argument of G ; we do not need to explicitly specify the $\hat{\alpha}_i$'s (nor even show the existence of $\hat{\alpha}_i$'s satisfying the specified $\hat{\alpha}_{ij}$'s). For four or more arguments, G satisfies the recursion relation

$$G(\hat{\alpha}_1, \dots, \hat{\alpha}_m) = (-1)^{1+\Theta(\hat{\alpha}_{m-1,m})} \Theta \left(-\hat{\alpha}_{m-1,m} \sum_{i=1}^{m-1} \hat{\alpha}_{im} \right) G(\bar{\alpha}_1, \dots, \bar{\alpha}_{m-1}) + \sum_B \Delta G_B, \quad (2.16)$$

where $\bar{\alpha}_i$'s for $1 \leq i \leq (m-1)$ satisfy

$$\begin{aligned} \bar{\alpha}_{m-3,m-1} &= -\bar{\alpha}_{m-1,m-3} = \hat{\alpha}_{m-3,m-1} + \sum_{i=1}^{m-1} \hat{\alpha}_{im}, \\ \bar{\alpha}_{ij} &= \hat{\alpha}_{ij} \quad \text{for } (i, j) \neq (m-1, m-3) \text{ or } (m-3, m-1). \end{aligned} \quad (2.17)$$

The sum over B in (2.16) runs over all subsets of $\hat{\alpha}_1, \dots, \hat{\alpha}_m$ of three or more *consecutive elements* and containing either $\hat{\alpha}_m$, or both $\hat{\alpha}_{m-1}$ and $\hat{\alpha}_{m-3}$ (or all three of them). ΔG_B is given as follows. First of all we introduce auxiliary elements $\hat{\alpha}'_1, \dots, \hat{\alpha}'_m$ of \mathcal{V} satisfying

$$\begin{aligned} \hat{\alpha}'_{im} &= -\hat{\alpha}'_{mi} = \mu_B \hat{\alpha}_{im} \quad \text{for } i = 1, 2, \dots, m-1, \\ \hat{\alpha}'_{m-3,m-1} &= -\hat{\alpha}'_{m-1,m-3} = \hat{\alpha}_{m-3,m-1} + (1 - \mu_B) \sum_{i=1}^{m-1} \hat{\alpha}_{im}, \\ \hat{\alpha}'_{ij} &= \hat{\alpha}_{ij} \quad \text{for all other } i, j, \end{aligned} \quad (2.18)$$

where μ_B takes different values for different choice of B and will be specified below.

1. For $B = \{\hat{\alpha}_{m-2}, \hat{\alpha}_{m-1}, \hat{\alpha}_m\}$ and $m > 4$, we have

$$\mu_B = -\hat{\alpha}_{m-2,m-1} / (\hat{\alpha}_{m-1,m} + \hat{\alpha}_{m-2,m}), \quad (2.19)$$

and

$$\begin{aligned} \Delta G_B &= G(\hat{\alpha}'_1, \dots, \hat{\alpha}'_{m-3}, \hat{\alpha}'_{m-2} + \hat{\alpha}'_{m-1} + \hat{\alpha}'_m) \times G(\hat{\alpha}'_{m-2}, \hat{\alpha}'_{m-1}, \hat{\alpha}'_m) \\ &\quad \times \text{sign}(\hat{\alpha}_{m-1,m} + \hat{\alpha}_{m-2,m}) \Theta(-(\hat{\alpha}_{m-2,m-1} + \hat{\alpha}_{m-1,m} + \hat{\alpha}_{m-2,m}) \hat{\alpha}_{m-2,m-1}) . \end{aligned} \quad (2.20)$$

2. For $B = \{\hat{\alpha}_k, \dots, \hat{\alpha}_m\}$ for $3 \leq k \leq (m-3)$ we have

$$\mu_B = \left(\sum_{\substack{i,j \\ k \leq i < j \leq m-1}} \hat{\alpha}_{ij} + \sum_{i=1}^{m-1} \hat{\alpha}_{im} \right) / \sum_{i=1}^{k-1} \hat{\alpha}_{im} , \quad (2.21)$$

and

$$\begin{aligned} \Delta G_B &= G\left(\hat{\alpha}'_1, \dots, \hat{\alpha}'_{k-1}, \sum_{i=k}^m \hat{\alpha}'_i\right) \times G(\hat{\alpha}'_k, \hat{\alpha}'_{k+1}, \dots, \hat{\alpha}'_m) \text{sign}\left(-\sum_{i=1}^{k-1} \hat{\alpha}_{im}\right) \\ &\quad \times \Theta\left(-\left(\sum_{\substack{i,j \\ k \leq i < j \leq m}} \hat{\alpha}_{ij}\right) \left(\sum_{\substack{i,j \\ k \leq i < j \leq m-1}} \hat{\alpha}_{ij} + \sum_{i=1}^{m-1} \hat{\alpha}_{im}\right)\right) . \end{aligned} \quad (2.22)$$

3. For $B = \{\hat{\alpha}_k, \dots, \hat{\alpha}_{m-1}\}$ for $2 \leq k \leq (m-3)$ we have

$$\mu_B = \left(\sum_{\substack{i,j \\ k \leq i < j \leq m-1}} \hat{\alpha}_{ij} + \sum_{i=1}^{m-1} \hat{\alpha}_{im} \right) / \sum_{i=1}^{m-1} \hat{\alpha}_{im} , \quad (2.23)$$

and

$$\begin{aligned} \Delta G_B &= G\left(\hat{\alpha}'_1, \dots, \hat{\alpha}'_{k-1}, \sum_{i=k}^{m-1} \hat{\alpha}'_i, \hat{\alpha}'_m\right) \times G(\hat{\alpha}'_k, \hat{\alpha}'_{k+1}, \dots, \hat{\alpha}'_{m-1}) \text{sign}\left(-\sum_{i=1}^{m-1} \hat{\alpha}_{im}\right) \\ &\quad \times \Theta\left(-\left(\sum_{\substack{i,j \\ k \leq i < j \leq m-1}} \hat{\alpha}_{ij} + \sum_{i=1}^{m-1} \hat{\alpha}_{im}\right) \left(\sum_{\substack{i,j \\ k \leq i < j \leq m-1}} \hat{\alpha}_{ij}\right)\right) . \end{aligned} \quad (2.24)$$

4. For $B = \{\hat{\alpha}_2, \dots, \hat{\alpha}_m\}$ and $m > 4$ we have

$$\mu_B = \left(\sum_{\substack{i,j \\ 2 \leq i < j \leq m}} \hat{\alpha}_{ij} + \hat{\alpha}_{1m} \right) / \hat{\alpha}_{1m} , \quad (2.25)$$

and

$$\Delta G_B = \Theta \left(- \left(\sum_{\substack{i,j \\ 2 \leq i < j \leq m}} \hat{\alpha}_{ij} \right) \left(\sum_{\substack{i,j \\ 2 \leq i < j \leq m}} \hat{\alpha}_{ij} + \hat{\alpha}_{1m} \right) \right) \text{sign}(-\hat{\alpha}_{1m}) G(\hat{\alpha}'_2, \dots, \hat{\alpha}'_m). \quad (2.26)$$

5. For $m = 4$ and $B = \{\hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4\}$ we have

$$\mu_B = -\hat{\alpha}_{23}/(\hat{\alpha}_{24} + \hat{\alpha}_{34}), \quad (2.27)$$

and

$$\Delta G_B = \Theta(-\hat{\alpha}_{23}(\hat{\alpha}_{23} + \hat{\alpha}_{34} + \hat{\alpha}_{24})) \times \text{sign}(\hat{\alpha}_{24} + \hat{\alpha}_{34}) G(\hat{\alpha}'_2, \hat{\alpha}'_3, \hat{\alpha}'_4). \quad (2.28)$$

This finishes our description of ΔG_B in all cases. This in turn defines G , and hence F and g_C . The equivalence of the definitions of g_C given in §2.2.1 and in this section was proven in [11].

2.3 Definition of the functions H

We now give the algorithm for constructing the functions $H(\{\beta_i\}; \{k_i\}; y)$ appearing in (2.4).

1. When the number of β_i 's is less than three, $H(\{\beta_i\}; \{k_i\}; y)$ vanishes.
2. For three or more number of β_i 's, observe that the expression for $Q_C(\sum_i k_i \beta_i; \zeta; y)$ given in (1.7), (2.2) contains a term proportional to $H(\{\beta_i\}; \{k_i\}; y) \prod_i \Omega_S(\beta_i; t^{k_i})$ arising from the choice $m = 1$ in (1.7), $n = 1$, $\alpha_1 = \sum_i k_i \beta_i$, $m_1 = 1$ in (2.2), and $m_i = k_i$ in the expression for $\Omega_{\text{tot}}(\sum_i k_i \beta_i; y, t)$ in eq.(2.4). We fix $H(\{\beta_i\}; \{k_i\}; y)$ by demanding that the net coefficient of the product $\prod_i \Omega_S(\beta_i; t^{k_i})$ in the expression for $Q_C(\sum_i k_i \beta_i; y, t)$ is a Laurent polynomial in y , i.e. a finite sum of the form $\sum c_s y^s$ where c_s 's are constant and the sum over s runs over a finite set of positive and negative integers (including zero). This of course leaves open the possibility of adding to H a Laurent polynomial. This is resolved by using the *minimal modification hypothesis*, which requires that H must be symmetric under $y \rightarrow y^{-1}$ and vanish as $y \rightarrow \infty$ [9]. We determine $H(\{\beta_i\}; \{m_i\}; y)$ iteratively by beginning with the H 's with three β_i 's and then determining successively the H 's with more β_i 's.

For illustration, suppose that the coefficient of $\prod_i \Omega_S(\beta_i; t^{k_i})$ in some example was $(y^2 + y^{-2})/(y - y^{-1})^2$ before taking into account the $H(\{\beta_i\}; \{k_i\}; y) \prod_i \Omega_S(\beta_i; t^{k_i})$ term. Then we choose $H(\{\beta_i\}; \{k_i\}; y) = -2/(y - y^{-1})^2$ so that the total coefficient of $\prod_i \Omega_S(\beta_i; t^{k_i})$ becomes $(y^2 + y^{-2} - 2)/(y - y^{-1})^2 = 1$, which is a Laurent polynomial in y .

3. H is expected to be independent of the FI parameters and hence can be calculated for any value of these parameters.

A useful property of $H(\{\beta_1, \dots, \beta_n\}; \{k_1, \dots, k_n\}; y)$ is that it vanishes if the set $\{1, \dots, n\}$ can be divided into two sets A and B such that $\langle \beta_i, \beta_j \rangle \geq 0$ for all $i \in A$ and $j \in B$ [15].

2.4 Constraints on $\Omega_S(\alpha; t)$'s

In the Coulomb branch formula $\Omega_S(\gamma; t)$ are unknown functions of t . In this section we shall discuss some necessary conditions on γ in order for $\Omega_S(\gamma; t)$ *not* to vanish. It is important to remember, however, that these conditions should be used *after determining the functions* $H(\{\beta_i\}; \{k_i\}; y)$ *following the procedure of §2.3 where we treat all the $\Omega_S(\gamma; t)$ for $\gamma \in \Gamma^+$ to be non-vanishing and independent of each other.*

The first set of conditions on $\Omega_S(\gamma; t)$ are

$$\Omega_S(a\gamma_i + b\gamma_j; t) = \begin{cases} 1 & \text{if } a = 1, b = 0 \text{ or } a = 0, b = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.29)$$

for any linear combination of two basis vectors $a\gamma_i + b\gamma_j$. $\Omega_S(\sum_i n_i \gamma_i; t)$ can be non-zero if at least 3 of the n_i 's are non-zero, but only if the following condition holds. Let β_1, \dots, β_n be the set of $n = \sum_i n_i$ vectors with n_1 of the β_k 's being γ_1 , n_2 of the β_k 's being γ_2 etc., and let $\beta_{k\ell} \equiv \langle \beta_k, \beta_\ell \rangle$. Then for $\Omega_S(\sum_i n_i \gamma_i; t)$ to be non-zero there must exist n vectors $\vec{r}_1, \dots, \vec{r}_n \in \mathbb{R}^3$ satisfying the equations

$$\sum_{\ell=1}^n \frac{\beta_{k\ell}}{|\vec{r}_k - \vec{r}_\ell|} = 0 \quad \forall \quad k. \quad (2.30)$$

For $n = 3$ this condition requires that the set of numbers $(\beta_{12}, \beta_{23}, \beta_{31})$ or $(-\beta_{12}, -\beta_{23}, -\beta_{31})$ are all positive and satisfy triangle inequality [16]. For $n \geq 4$ there is no such simple criteria but one can show that (2.30) has no solutions if we can divide the set $\{1, \dots, n\}$ into two sets A and B such that $\beta_{k\ell} \geq 0$ for all $k \in A$ and $\ell \in B$ [15].

Further constraints on Ω_S can be obtained using mutation symmetry of the quiver. This has been discussed in (3.16).

The Coulomb branch formula gives an expression for the Dolbeault polynomial – a function of two variables y and t – in terms of $\Omega_S(\alpha; t)$'s which are functions of a single variable t . Furthermore the Dolbeault polynomial depends on the FI parameters ζ but the $\Omega_S(\alpha; t)$'s do not. So by knowing $\Omega_S(\alpha; t)$ for one ζ one can deduce $Q_C(\gamma; \zeta; y, t)$ for any other values of the FI parameters. A practical way of computing $\Omega_S(\alpha; t)$ is to specialize (2.2), to $y = 1/t = v^{1/2}$, where the left-hand side becomes proportional to the Hirzebruch polynomial $\chi(\mathcal{M}(\gamma; \zeta), v)$

via eq.(1.6). Inverting this relation allows to express $\Omega_S(\alpha; t)$ in terms of the Hirzebruch polynomials $\chi(\mathcal{M}(\gamma; \zeta), v)$ associated to subquivers, for a fixed choice of stability condition ζ .⁴

2.5 Deformations of the input parameters

As mentioned in §2.2.1 and §2.2.2, in order to determine g_C unambiguously we need to work with a deformed set of parameters. In this subsection we shall review this prescription. However we would like to mention at the outset that while the deformed parameters are to be used in eqs.(2.8) and (2.9) for enumerating the solutions and computing $s(P)$, and also computing the step and sign functions appearing in various equations in §2.2.2, the exponent of y in (2.7) or (2.10) is always computed with the undeformed α_{ij} 's, related to the undeformed γ_{ij} 's via (1.2).

1. In the first step we deform the original numbers γ_{ij} by arbitrary small numbers:

$$\gamma_{ij} \rightarrow \gamma_{ij} + \epsilon_1 \xi_{ij}, \quad (2.31)$$

where ϵ_1 is a small positive number and ξ_{ij} are arbitrary but sufficiently generic real numbers between -1 and 1 satisfying $\xi_{ij} = -\xi_{ji}$. In particular for given ϵ_1 , ξ_{ij} should be such that any ordered set of vectors $\beta_1, \dots, \beta_s \in \Gamma^+$ with $s \geq 2$ and not all β_k 's parallel, satisfying $\sum_k \beta_k = \sum_i n_i \gamma_i$ with $n_i \leq N_i$, have the property that $\sum_{i < j} \beta_{ij} \neq 0$. This condition ensures that the set of equations (2.8) has no solutions where two or more centers coincide, except if the α_i 's associated with all the centers are parallel. Furthermore ϵ_1 should be sufficiently small so that for given ξ_{ij} the above condition holds for any ϵ_1 between the chosen value and 0 except possibly at $\epsilon_1 = 0$. Since the undeformed β_{ij} 's are all integers, the last condition can be ensured by taking ϵ_1 to be less than $2/N(N-1)$ where $N = \sum_i N_i$ is the rank of the quiver.

The deformation parameters ϵ_1 and ξ_{ij} , once chosen, must be kept fixed throughout the calculation, *e.g.* for g_C with various arguments as well as for calculating the functions H using the procedure described in §2.3 we must use the same deformation parameters.

2. The above deformation does not completely remove the ambiguity in enumerating the solutions to (2.8) since two or more z_i 's whose associated α_i 's are parallel can still coincide. Furthermore if the original vector γ is not primitive then for the arguments c_i of g_C in (2.2), $\sum_i' c_i$ may vanish where \sum_i' denotes sum over a proper subset of the elements. This

⁴Note that the Coulomb branch formula expresses $Q_C(\gamma; \zeta; y, t)$ for $\gamma = \sum_i N_i \gamma_i$ in terms of $\Omega_S(\alpha; t)$ for $\alpha = \sum_i p_i \gamma_i$, $0 \leq p_i \leq N_i$. Thus to systematically calculate the Ω_S 's from the knowledge of χ we need to begin with the lowest rank quivers (minimum possible values of $\sum_i N_i$) and then work our way upwards.

allows a solution where two sets of z_i 's have infinite separation from each other. To avoid such situations we carry out a second deformation in which we choose some arbitrary ordering of the α_i 's and deform the α_{ij} 's and c_i 's in the argument of g_C to

$$\alpha_{ij} \rightarrow \alpha_{ij} + \epsilon_2 \eta_{ij}, \quad c_i \rightarrow c_i + \epsilon_2 f_i. \quad (2.32)$$

Here ϵ_2 is a small positive number, and f_i and η_{ij} 's are arbitrary but sufficiently generic real numbers between -1 and 1 satisfying $\eta_{ij} = -\eta_{ji} > 0$ for $j > i$ and $\sum_i f_i = 0$. For given ϵ_1 , ξ_{ij} and ϵ_2 , η_{ij} and f_i should be such that $\sum_{i,j \in A; i < j} \alpha_{\sigma(i)\sigma(j)}$ is non-zero for any permutation σ and any subset A of $1, \dots, n$ containing two or more elements, and $\sum_{i \in B} c_i$ should be non-zero for any non-empty proper subset B of $1, \dots, n$. These conditions ensure that with the deformed parameters the set of equations (2.8) has no solutions where two or more centers coincide or where one or more centers get infinitely separated from the rest. Furthermore ϵ_2 should be sufficiently small so that for given ϵ_1 , ξ_{ij} and η_{ij} , the conditions mentioned above hold for all ϵ_2 between the chosen value and 0 , except possibly at $\epsilon_2 = 0$. If we choose the original values of ζ_i 's and the deformations $\epsilon_1 \xi_{ij}$ to be integer multiples of $1/\Lambda$ for some large number Λ , then the last condition can be ensured by taking ϵ_2 to be less than $2/(n(n-1)\Lambda)$.

Note that this set of deformations are ‘local’ *e.g.* specific to the evaluation of g_C with a specific set of arguments. For evaluating g_C with another set of arguments we use another set of deformation parameters ϵ_2 and η_{ij} satisfying the conditions described above. However different terms contributing to a given g_C must be computed with the same set of deformation parameters.

3. This finishes the procedure that needs to be followed for the evaluation of g_C given in §2.2.1. However this still leaves some ambiguity in the analysis of §2.2.2 since even after we deform the parameters α_{ij} and c_i using the two step process described above, one or more of the arguments of the step and sign functions on the right hand side of (2.12), and similar functions appearing in the recursion relations for G , may vanish accidentally making these functions ambiguous. These occur on codimension ≥ 1 subspaces of the space of the deformation parameters ϵ_2 and α_{ij} and we can adjust these parameters to keep away from these subspaces.

It was argued in §3.2 of [11] that Q_C computed from this procedure is independent of the choice of the deformation parameters $\epsilon_1 \xi_{ij}$ and $\epsilon_2 \eta_{ij}$.

For quivers without oriented loops an alternative version of the Coulomb branch formula that does not require the deformations given in (2.31), (2.32) was written down in [17]. We do

not know of an extension of this to the general case.

2.6 Mathematica code

The algorithm for computing the Dolbeault polynomial Q_C using the Coulomb branch formula was implemented in a Mathematica package called `CoulombHiggs.m`, whose first version was described in Appendix A of [11]. The latest release and updated instructions can be downloaded from

<http://www.lpthe.jussieu.fr/~pioline/computing.html>

The package contains many other commands to compute e.g. the Coulomb indices g_C or the Reineke formula mentioned in §3.1 of this review. Here we describe the main command for computing Coulomb branch formula. After copying the file `CoulombHiggs.m` in the same directory as the notebook, evaluate

```
In[1]:= SetDirectory[NotebookDirectory[]]; <<CoulombHiggs.m
Out[1]:= CoulombHiggs v2.1 - A package for evaluating quiver
          invariants using the Coulomb and Higgs branch formulae.
```

The main routine for evaluating Q_C is `CoulombBranchFormula`, whose usage is as follows:

```
In[2]:= Simplify[CoulombBranchFormula[4{{0, 1, -1},{-1, 0, 1}, {1,
-1, 0}}, {1/2, 1/6, -2/3}, {1, 1, 1}]]
Out[2]:= 2 +  $\frac{1}{y^2}$  +  $y^2$  + OmS({1, 1, 1}, y, t)
```

This routine computes $Q_C(\gamma; \zeta; y, t)$. The first argument corresponds to the matrix of DSZ products γ_{ij} (an antisymmetric matrix of integers), the second to the FI parameters ζ_i (a vector of rational numbers) and the third to the dimension vector N_i (a vector of integers). The code allows for the possibility of y -dependence in Ω_S (called OmS in the code), but for generic superpotential the single-centered indices $\Omega_S(\gamma; y, t)$ can be taken to be independent of y . The constraint (2.30) has not been incorporated in the code, so some of the Ω_S 's appearing in the expression for Q_C may actually be zero.

One of the bottlenecks in running the code is due to the need for the deformations described in (2.31) and (2.32). The current version uses randomly generated small rational numbers for these deformations. This works well for small number of nodes, but when the number of nodes becomes large (say ≥ 6) occasionally the arguments of one of the step functions vanish accidentally. In that case, the step function evaluates to 1/2 and generates a warning message.

In such cases it is advisable to run the code again: since each run uses different random deformations, typically the warning messages will disappear.

Irrespective of this issue, the code could certainly be optimized by compiling and/or parallelizing the computation of the Coulomb indices (the performance of the version 2.1 released along with this review is already greatly improved with respect to previous versions, by avoiding to evaluate Coulomb indices which end up being multiplied by zero).

3 Aspects of the Coulomb branch formula

In this section we shall discuss various aspects of the Coulomb branch formula.

3.1 Quivers without oriented loops and Reineke's formula

For quivers without oriented loops, the necessary condition (2.30) always fails and hence all $\Omega_S(\gamma)$'s vanish except for the $\Omega_S(\gamma_i)$'s associated to basis vectors γ_i , which are all equal to 1 according to (2.29). Thus Q_C computed from (2.2) becomes t -independent. Furthermore by suitably choosing the deformation parameters described in §2.5 one can simplify the formula for g_C given in §2.2.2. The deformations are chosen as follows. First we choose the ξ_{ij} 's in (2.31) such that the deformed quiver also does not have oriented loop. In this case the α_i 's appearing in the argument of g_C in (2.2) have the property that they can be ordered such that $\alpha_{ij} \geq 0$ for $i < j$. We choose this ordering and then deform them as in (2.32) with $\eta_{ij} > 0$ for $i < j$ so that the deformed α_{ij} 's are all positive for $i < j$. One finds that in this case the functions G appearing in the right hand side of (2.12) all vanish due to the property mentioned in the paragraph below (2.14) and the recursion relation for F reduces to just the first term on the right hand side of (2.12). This gives⁵

$$F(\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_n\}; \{\tilde{c}_1, \dots, \tilde{c}_n\}) = \prod_{k=2}^n \left\{ \Theta(-\tilde{\alpha}_{k-1,k}(\tilde{c}_k + \dots \tilde{c}_n)) (-1)^{\Theta(-\tilde{\alpha}_{k-1,k})} \right\}. \quad (3.1)$$

Furthermore all the relevant H -functions also vanish due to the property mentioned at the end of §2.3. It was shown in §4 of [11] that the Coulomb branch formula computed with this expression for F reduces to Reineke's formula [18] for the Poincaré polynomial of the moduli space of quivers without oriented loops.

⁵If we had chosen a different set of deformation parameters then the intermediate steps will be more complicated but the final results will be the same.

3.2 Wall-crossing

It was shown in §4.1 and §5.4 of [9] that the Coulomb branch formula satisfies the wall-crossing formula given in [14] (and further elaborated in [19–21]), provided the single-centered invariants $\Omega_S(\gamma)$ stay constant across the wall.

In the context of quivers the wall-crossing formula can be stated as follows. First note that (1.1) gives $\zeta \cdot \gamma = 0$ for $\gamma = \sum_i N_i \gamma_i$. A quiver carrying charge vector γ hits a wall of marginal stability if the FI parameters take values such that $\zeta \cdot \gamma' = 0$ for another vector $\gamma' \not\parallel \gamma$. If we denote by $\tilde{\Gamma}^+$ the intersection of Γ^+ with the plane spanned by γ and γ' then it follows that for any vector $\alpha \in \tilde{\Gamma}^+$, $\zeta \cdot \alpha$ vanishes on the wall. Let ζ^L, ζ^R denote two different points in the FI parameter space on two sides of the marginal stability wall, satisfying the conditions (1.1) and:

$$\langle \beta, \gamma \rangle (\zeta^L \cdot \beta) > 0, \quad \langle \beta, \gamma \rangle (\zeta^R \cdot \beta) < 0, \quad \forall \beta \in \tilde{\Gamma}^+. \quad (3.2)$$

For any vector $\alpha \in \tilde{\Gamma}^+$ let us define

$$\zeta_\alpha \equiv \zeta - (\langle (\gamma - \alpha), \gamma_1 \rangle, \dots, \langle (\gamma - \alpha), \gamma_K \rangle) (\langle \gamma, \alpha \rangle)^{-1} \zeta \cdot \alpha. \quad (3.3)$$

Note that $\zeta_\alpha \cdot \alpha = 0$ automatically. For ζ close to the wall of marginal stability, ζ_α is close to ζ since $\zeta \cdot \alpha \simeq 0$. Furthermore ζ_α is on the ‘same side’ of the wall as ζ in the sense that (3.2) holds with γ replaced by α and ζ replaced by ζ_α . With this definition of ζ_α the wall-crossing formula of [14] can be stated as⁶

$$\bar{Q}_C(\gamma, \zeta^L; y, t) = \sum_n \frac{1}{n!} \sum_{\substack{\alpha_i \in \tilde{\Gamma}^+ \\ \sum_i \alpha_i = \gamma}} g_C(\alpha_1, \dots, \alpha_n; c_1^R, \dots, c_n^R; y) \prod_{i=1}^n \bar{Q}_C(\alpha_i, \zeta_{\alpha_i}^R; y, t), \quad (3.4)$$

where

$$c_i^R = \zeta^R \cdot \alpha_i. \quad (3.5)$$

The sum over $\{\alpha_i\}$ in (3.4) runs over all *ordered* set of vectors $\alpha_1, \dots, \alpha_n \in \tilde{\Gamma}^+$ satisfying $\sum_i \alpha_i = \gamma$. Furthermore, even if the original quiver γ has oriented loops, the α_i ’s appearing in the argument of g_C on the right hand side of (3.4) can be partially ordered so that $\alpha_{ij} \geq 0$ for $i < j$. Hence by deforming them so that $\alpha_{ij} > 0$ for $i < j$, we can use (2.10) with the expression for F given in (3.1) for computing g_C . The equivalence of this wall-crossing formula and the Kontsevich-Soibelman wall-crossing formula [8] was proven in [17, 22].

⁶The definition of ζ_α was not given explicitly before, but it follows from the analysis in §4.1 of [9] in a straightforward manner.

3.3 Generalized quiver mutations

The Coulomb branch formula (2.2) is conjectured to be invariant under a general mutation symmetry [26]. To describe this conjecture in its most general form we need a slight generalization of this formula as follows:

1. We take Ω_S to depend on y and denote them by $\Omega_S(\gamma; y, t)$. In all the formulæ in §2.1 we replace $\Omega_S(\gamma; t^m)$ by $\Omega_S(\gamma; y^m, t^m)$.
2. We relax the constraints (2.29) and (2.30).

We now pick a particular node k and define $\Omega_{n,s}$ and M via

$$\Omega_S(\ell\gamma_k; y, t) = \sum_{n,s} \Omega_{n,s}(\ell\gamma_k) y^n t^s, \quad M \equiv \sum_{\ell \geq 1} \sum_{n,s} \ell^2 \Omega_{n,s}(\ell\gamma_k). \quad (3.6)$$

We further define

$$\begin{aligned} \gamma'_i &= \begin{cases} -\gamma_k & \text{if } i = k \\ \gamma_i + M \max(0, \varepsilon \gamma_{ik}) \gamma_k & \text{if } i \neq k \end{cases} \\ \gamma'_{ij} &= \begin{cases} -\gamma_{ij} & \text{if } i = k \text{ or } j = k \\ \gamma_{ij} + M \max(0, \gamma_{ik} \gamma_{kj}) \text{sign}(\gamma_{kj}) & \text{if } i, j \neq k \end{cases} \\ \zeta'_i &= \begin{cases} -\zeta_k & \text{if } i = k \\ \zeta_i + M \max(0, \varepsilon \gamma_{ik}) \zeta_k & \text{for } i \neq k, \end{cases} \\ N'_i &= \begin{cases} -N_k + M \sum_{j \neq k} N_j \max(0, \varepsilon \gamma_{jk}) & \text{if } i = k \\ N_i & \text{if } i \neq k \end{cases} \end{aligned} \quad (3.7)$$

where ε takes values ± 1 with the $+$ sign corresponding to right mutation and $-$ sign corresponding to left mutation. These transformation laws, which will be called the generalized mutation transformations, guarantee that

$$\gamma \equiv \sum_i N_i \gamma_i = \sum_i N'_i \gamma'_i. \quad (3.8)$$

We also define $\Omega'_S(\alpha; y, t)$ through the relations

$$\Omega_S(\alpha; y, t) = \begin{cases} \Omega'_S(\alpha + M \max(0, \varepsilon \langle \alpha, \gamma_k \rangle) \gamma_k; y, t) & \text{for } \alpha \not\parallel \gamma_k \\ \Omega'_S(-\alpha; y, t) & \text{for } \alpha \parallel \gamma_k \end{cases}. \quad (3.9)$$

In [26] we conjectured that $Q_C(\gamma; \zeta; y, t)$ produced by the Coulomb branch formula is invariant under the generalized mutation transformation:

$$Q_C(\gamma; \zeta; y, t) = \begin{cases} Q'_C(\gamma; \zeta'; y, t) & \text{if } \gamma \not\parallel \gamma_k \\ Q'_C(-\gamma; \zeta; y, t) & \text{if } \gamma \parallel \gamma_k, \end{cases} \quad (3.10)$$

provided the following conditions hold:

$$\varepsilon \zeta_k < 0, \quad (3.11)$$

$$\Omega_{n,s}(\ell \gamma_k) \geq 0 \ \forall \ \ell > 0, \quad \Omega_{n,s}(\ell \gamma_k) = 0 \text{ for } \ell > \ell_{\text{Max}}, \quad (3.12)$$

for some ℓ_{Max} . In (3.10), it is understood that in computing the l.h.s. we have to express γ as $\sum_i N_i \gamma_i$ treating γ_i 's as the basis vectors and apply the Coulomb branch formula (2.2), (2.4) with single centered indices $\Omega_S(\alpha; y, t)$, while in computing the r.h.s. we have to express γ as $\sum_i N'_i \gamma'_i$ treating γ'_i 's as the basis vectors and then apply the Coulomb branch formula (2.2), (2.4) with single centered indices $\Omega'_S(\alpha; y, t)$. Since $\Omega'_S(\alpha'; y, t) = 0$ unless $\alpha' = \sum_i n''_i \gamma'_i$ with $n''_i \geq 0$, the transformation law (3.9) forces the corresponding $\Omega_S(\alpha; y, t)$'s also to vanish. This can be made explicit by expressing (3.9) as

$$\Omega_S \left(\sum_i n_i \gamma_i \right) = \Omega_S \left(\sum_i n''_i \gamma'_i \right), \quad (3.13)$$

where

$$\begin{aligned} n''_i &= n_i \quad \text{for } i \neq k, \\ n''_k &= -n_k + M \sum_{j \neq k} n_j \max(0, \varepsilon \gamma_{jk}) - M \max \left(0, \varepsilon \sum_{j \neq k} n_j \gamma_{jk} \right). \end{aligned} \quad (3.14)$$

Thus the condition $n''_k \geq 0$ for non-vanishing Ω'_S translates to the following condition for getting non-vanishing $\Omega_S(\sum_i n_i \gamma_i)$:

$$n_k \leq M \sum_{j \neq k} n_j \max(0, \varepsilon \gamma_{jk}) - M \max \left(0, \varepsilon \sum_{j \neq k} n_j \gamma_{jk} \right). \quad (3.15)$$

For $M = 1$ these transformation rules reduce to the usual mutation transformation rules discussed in [8, 27–29]. If we further restrict to quivers with generic superpotentials where Ω_S 's become y independent, then (3.9) shows that Ω'_S 's are also y -independent. This is consistent with the fact that under mutation a generic superpotential transforms to a generic superpotential. Note however that the fact that the Coulomb branch formula with y -independent

Ω_S 's transforms under mutation to the Coulomb branch formula with y -independent Ω_S 's is a non-trivial property of the Coulomb branch formula. Thus a test of (generalized) mutation symmetry of the Coulomb branch formula provides us with an indirect test of the Coulomb branch formula itself. Furthermore since both left and right mutations are known to be symmetries of the quiver, (3.15) now gives a constraint on the Ω_S 's associated with the physical quivers. We can in fact make these constraints stronger by requiring Ω'_S to satisfy (2.29). This gives the following condition on $\Omega_S(\alpha)$:

$$\begin{aligned}\Omega_S \left(\sum_i n_i \gamma_i \right) &= 0 \text{ unless } n_k \leq \sum_{j \neq k} n_j \max(0, \varepsilon \gamma_{jk}) - \max \left(0, \varepsilon \sum_{j \neq k} n_j \gamma_{jk} \right) \quad \forall k; \varepsilon = \pm 1, \\ \Omega_S(a\gamma_i + b\gamma_j + c\gamma_k) &= 0 \text{ unless } c < a \max(0, \varepsilon \gamma_{ik}) + b \max(0, \varepsilon \gamma_{jk}) - \max(0, \varepsilon(a\gamma_{ik} + b\gamma_{jk})) \\ &\quad \forall \text{ distinct } i, j, k; a, b > 0; \varepsilon = \pm 1.\end{aligned}\tag{3.16}$$

3.4 Other tests

The Coulomb branch formula passes several other consistency tests.

1. It was shown in [10, 23] that the Coulomb branch formula correctly reproduces the cohomology of cyclic abelian quivers which was calculated directly from the analysis of (1.4) in [24, 25].
2. The Coulomb branch formula was tested in many other explicit examples in §5 and §6 of [10].
3. For quivers with oriented loops the Reineke formula given in [18], called the Higgs branch formula in [11], no longer gives the result for the Poincaré polynomial. However one can regard this formula as the result for the cohomology of the (non-compact) space obtained by imposing only the D-term constraints given by the first set of equations in (1.4), together with the effect of taking the quotient by the $\prod_i U(N_i)$ gauge groups. Let us call this the embedding space. Under suitable conditions the F-term constraints, given by the second set of equations in (1.4), can be regarded as the condition for vanishing section of line bundles with positive curvature. Using repeated application of Lefschetz hyperplane theorem we can then relate the Hodge numbers $h^{p,q}$ of the quiver moduli space \mathcal{M} to the Hodge number of the embedding space for low values of $(p + q)$. Since the former is conjecturally given by the Coulomb branch formula whereas the latter is given by the Higgs branch formula, this gives a specific relation between the Coulomb and the Higgs branch formulæ. The precise relation takes the following form [15]. If we

take the expression for $Q_C(\gamma; \zeta; y, t)$ then the coefficients of the powers of y^{-1} , that do not have any explicit dependence on unknown Ω_S 's after using (2.29), must match with the corresponding coefficients in the Higgs branch formula. It is however important that *we do not use (2.30) to set some of the Ω_S 's to zero for making this comparison, although we are allowed to set $\Omega_S(\sum_i n_i \gamma_i; t)$ to zero if the subquiver given by the dimension vector (n_1, \dots, n_K) does not contain a closed loop.* The technical reason for this is that even if a subquiver fails to satisfy (2.30) but has oriented loop, we can construct a gauge invariant superpotential involving the variables $\phi_{\ell k; \alpha; st}$ associated with the subquiver. This correspondence between the Coulomb and the Higgs branch formula has been tested in a number of examples [15].

3.5 Variants of the conjecture

The version of the conjecture stated in §2 is in its strongest form. There are however some more conservative versions of this conjecture which we describe below.

1. If we set $t = 1$ then the Dolbeault Polynomial (1.5) reduces to the Poincaré polynomial containing information about the Betti numbers only. The Coulomb branch formula given in (2.2) for $t = 1$ now gives information on the Betti numbers in terms of $\Omega_S(\gamma; t = 1)$.⁷ The latter in turn can be related to the Euler number χ of \mathcal{M} by setting $y = t = 1$ on both sides of (2.2) since in that case (1.5) gives $Q(\mathcal{M}; y = 1, t = 1) = (-1)^d \chi$. Thus in this case the Coulomb branch formula can be interpreted as an expression for the Betti numbers of \mathcal{M} in terms of the Euler number. This is the form in which the conjecture is directly related to black hole quantum mechanics and this motivated this conjecture in the first place. Thus a weaker form of the conjecture would be that it is valid only for $t = 1$ and not for general t .
2. Another version of the conjecture allows the Ω_S 's to depend both on t and y . In this case, in order to ensure consistency with the wall-crossing formula, all factors of $\Omega_S(\alpha; t^m)$ will have to be replaced by $\Omega_S(\alpha; y^m, t^m)$ in the Coulomb branch formula. Now for each quiver we have an unknown function $\Omega_S(\gamma; y, t)$ and hence if for each γ we consider a fixed set of FI parameters $(\zeta_1, \dots, \zeta_K)$, then the Coulomb branch formula has no information. However since $\Omega_S(\gamma; y, t)$ is independent of the FI parameters ζ , the Coulomb branch formula determines the Hodge numbers for all values of FI parameters in terms of the Hodge numbers at one point of FI parameter space. We can in fact regard such a formula

⁷Note that this has to be done after we have determined the functions H following the procedure described in §2.3.

as the most general solution to the wall-crossing formula, since it has as input exactly the right amount of data, i.e. one function of y and t for every γ , representing the results at one point in the space of FI parameters. Thus such a formula may be of more general applicability, *e.g.* for quivers with non-generic superpotentials.

3.6 Application to framed BPS indices

Beyond its utility for computing the cohomology of quiver moduli spaces, and applications to black hole counting, the Coulomb branch formula can also be used to compute framed BPS indices associated to line defects in $\mathcal{N} = 2$ supersymmetric field theories [30]. For a wide class of models, the framed BPS spectrum can be obtained by adjoining an additional ‘framing node’ γ_F to the original quiver that governs the BPS spectrum in the bulk. The prescription of [31] is that the framed BPS indices are computed using the Coulomb branch formula, with single-centered invariants $\Omega_S(\gamma)$ equal to 0 whenever the charge vector γ has non-zero support on γ_F , except for $\Omega_S(\gamma_F)$ itself, which is set equal to 1. It is checked in various examples that the resulting generating functions for framed invariants satisfy the Operator Product Expansion algebra for line defects [31] and agree for $y = 1$ with the generating functions computed in [30]. It is important to stress that the framed BPS indices obtained by this prescription differ from the BPS indices describing the cohomology of the corresponding framed quiver with generic superpotential, as the latter requires a different set of single centered invariants $\Omega_S(\gamma)$, which in general do not vanish when γ has support on the framing node.

Acknowledgments

We are grateful to Clay Cordova, Kentaro Hori, Andy Neitzke and Piljin Yi for valuable discussions. B.P. is grateful to HRI for hospitality during the final stage of this work. The work of A.S. was supported in part by the project 12-R&D-HRI-5.02-0303 and the J. C. Bose fellowship of the Department of Science and Technology, India.

References

- [1] M. R. Douglas and G. W. Moore, “D-branes, quivers, and ALE instantons,” hep-th/9603167.
- [2] F. Denef, “Quantum quivers and Hall / hole halos,” JHEP **0210** (2002) 023 [hep-th/0206072].
- [3] H. Derksen and J. Weyman, ”Quiver representations”, Notices Am. Math. Soc. **52** (2005) 200

- [4] M. Reineke, “Moduli of representations of quivers”, Proceedings of ICRA XII, Toruń, Poland, August 15–24, 2007, arXiv:0802.2147 [math-RT].
- [5] A. D. King, “Moduli of representations of finite dimensional algebras”, *Q. J. Math., Oxf. II. Ser.* **45** (1994) 515–530.
- [6] S. -J. Lee, Z. -L. Wang and P. Yi, “Abelianization of BPS Quivers and the Refined Higgs Index,” arXiv:1310.1265 [hep-th].
- [7] D. Joyce, “Configurations in Abelian categories. IV. Invariants and changing stability conditions”, [arXiv:math.AG/0410268].
- [8] M. Kontsevich and Y. Soibelman, “Stability structures, motivic Donaldson-Thomas invariants and cluster transformations”, [arXiv:0811.2435 [math.AG]].
- [9] J. Manschot, B. Pioline, and A. Sen, “A Fixed point formula for the index of multi-centered $N=2$ black holes,” *JHEP* **1105** (2011) 057, 1103.1887.
- [10] J. Manschot, B. Pioline, and A. Sen, “From Black Holes to Quivers,” *JHEP* **1211** (2012) 023, 1207.2230.
- [11] J. Manschot, B. Pioline, and A. Sen, “On the Coulomb and Higgs branch formulae for multi-centered black holes and quiver invariants,” *JHEP* **1305** (2013) 166, 1302.5498.
- [12] B. Pioline, “Corfu lectures on wall-crossing, multi-centered black holes, and quiver invariants,” *PoS Corfu* **2012** (2013) 085 [arXiv:1304.7159 [hep-th]].
- [13] J. Manschot, “Quivers and BPS bound states,” to appear.
- [14] J. Manschot, B. Pioline, and A. Sen, “Wall Crossing from Boltzmann Black Hole Halos,” *JHEP* **1107** (2011) 059, 1011.1258.
- [15] J. Manschot, B. Pioline and A. Sen, work in progress.
- [16] F. Denef and G. W. Moore, “Split states, entropy enigmas, holes and halos,” *JHEP* **1111**, 129 (2011) [hep-th/0702146 [HEP-TH]].
- [17] S. Mozgovoy, M. Reineke, “Abelian quiver invariants and marginal wall-crossing”, 1212.0410.
- [18] M. Reineke, “The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli,” *Invent. Math.* **152** (2003), no. 2, 349–368.
- [19] B. Pioline, “Four ways across the wall,” *J. Phys. Conf. Ser.* **346**, 012017 (2012) [arXiv:1103.0261 [hep-th]].
- [20] H. Kim, J. Park, Z. Wang and P. Yi, “Ab Initio Wall-Crossing,” *JHEP* **1109**, 079 (2011) [arXiv:1107.0723 [hep-th]].

- [21] M. Reineke, J. Stoppa, T. Weist, “MPS degeneration formula for quiver moduli and refined GW/Kronecker correspondence”, *Geom. Topol.* **16** (2012) 2097-2134, 1011.1258.
- [22] A. Sen, “Equivalence of Three Wall Crossing Formulae,” *Comm. Numb. Phys.* **6** (2012) 601-659, 1112.2515.
- [23] S.-J. Lee, Z.-L. Wang, and P. Yi, “BPS States, Refined Indices, and Quiver Invariants,” *JHEP* **1210** (2012) 094, 1207.0821.
- [24] I. Bena, M. Berkooz, J. de Boer, S. El-Showk, and D. Van den Bleeken, “Scaling BPS Solutions and pure-Higgs States,” *JHEP* **1211** (2012) 171, 1205.5023.
- [25] S.-J. Lee, Z.-L. Wang, and P. Yi, “Quiver Invariants from Intrinsic Higgs States,” *JHEP* **1207** (2012) 169, 1205.6511.
- [26] J. Manschot, B. Pioline and A. Sen, “Generalized quiver mutations and single-centered indices,” *JHEP* **1401** (2014) 050, 1309.7053.
- [27] H. Derksen, J. Weyman, and A. Zelevinsky, “Quivers with potentials and their representations. I: Mutations,” *Sel. Math., New Ser.* **14** (2008), no. 1, 59–119.
- [28] B. Keller and D. Yang, “Derived equivalences from mutations of quivers with potential,” *Adv. Math.* **226** (2011), no. 3, 2118–2168.
- [29] S. Mukhopadhyay and K. Ray, “Seiberg duality as derived equivalence for some quiver gauge theories,” *JHEP* **0402** (2004) 070, hep-th/0309191.
- [30] D. Gaiotto, G. W. Moore and A. Neitzke, “Framed BPS States,” arXiv:1006.0146 [hep-th].
- [31] C. Cordova and A. Neitzke, “Line Defects, Tropicalization, and Multi-Centered Quiver Quantum Mechanics,” arXiv:1308.6829 [hep-th].